

- b) Justify the steps which enable you to solve the quadratic equation  $2x^2 - 5x = 3$ .

$$2x^2 - 5x = 3$$

$$\Leftrightarrow 2x^2 - 5x - 3 = 0$$

$$\Leftrightarrow (2x + 1)(x - 3) = 0$$

$$\Leftrightarrow 2x + 1 = 0 \text{ or } x - 3 = 0$$

$$\Leftrightarrow x = -\frac{1}{2} \text{ or } x = 3$$

$$\text{Thus, } S = \left\{-\frac{1}{2}, 3\right\}.$$

*Subtract 3 from each side.*

*Factor the non-zero side.*

*Apply the zero product principle.*

*Solve each 1st degree equation.*

*Establish the solution set.*

## SOLVING QUADRATIC EQUATIONS BY FACTORING

- We call a **second degree equation** or **quadratic equation** in the variable  $x$  any equation that can be written in the form:  $ax^2 + bx + c = 0$ ,  $a \neq 0$ .

Ex.:  $2x^2 + 7x + 3 = 0$  ( $a = 2; b = 7; c = 3$ )  
 $3x^2 - 5x = 0$  ( $a = 3; b = -5; c = 0$ )  
 $x^2 - 4 = 0$  ( $a = 1; b = 0; c = -4$ )

are quadratic equations.

- The zero product principle makes it possible to solve quadratic equations by factoring.

- Write the equation in the general form:  
 $ax^2 + bx + c = 0$ .
- Factor the non-zero side.
- Apply the zero product principle.
- Solve each 1st degree equation.
- Write the solution set.

Ex.: Solve:  $x(5 - x) = 6$   
 $\Leftrightarrow x^2 - 5x + 6 = 0$   
 $\Leftrightarrow (x - 2)(x - 3) = 0$   
 $\Leftrightarrow x - 2 = 0 \text{ or } x - 3 = 0$   
 $x = 2 \text{ or } x = 3$   
 $S = \{2, 3\}$

2. Solve the following equations by factoring.

a)  $x^2 - 10x = 0$   $S = \{0, 10\}$

b)  $2x^2 + 5x = 0$   $S = \left\{0, -\frac{5}{2}\right\}$

c)  $3x^2 = 2x$   $S = \left\{0, \frac{2}{3}\right\}$

d)  $x^2 - 9 = 0$   $S = \{-3, 3\}$

e)  $2x^2 - 50 = 0$   $S = \{-5, 5\}$

f)  $4x^2 = 1$   $S = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

3. Solve the following equations by factoring.

a)  $x^2 - 2x - 15 = 0$   $S = \{5, -3\}$

b)  $x(x - 7) = -10$   $S = \{2, 5\}$

c)  $x^2 = 5x$   $S = \{0, 5\}$

d)  $4x(x - 3) = -9$   $S = \left\{\frac{3}{2}\right\}$

e)  $x(6x + 5) = 6$   $S = \left\{-\frac{3}{2}, \frac{2}{3}\right\}$

f)  $16x^2 = 25$   $S = \left\{-\frac{5}{4}, \frac{5}{4}\right\}$

## ACTIVITY 3 Form $x^2 = k$

a) Solve the equations

1.  $x^2 = 25$   $S = \{-5, 5\}$       2.  $x^2 = 7$   $S = \{-\sqrt{7}, \sqrt{7}\}$       3.  $x^2 = -1$   $S = \emptyset$

b) What are the solutions to the equation  $x^2 = k$  when

1.  $k > 0$ .  $S = \{-\sqrt{k}, \sqrt{k}\}$       2.  $k = 0$ .  $S = \{0\}$       3.  $k < 0$ .  $S = \emptyset$

### FORM $x^2 = k$

The equation  $x^2 = k$  has the solution set in  $\mathbb{R}$

- $S = \emptyset$  if  $k < 0$ .
- $S = \{0\}$  if  $k = 0$ .
- $S = \{-\sqrt{k}, \sqrt{k}\}$  if  $k > 0$ .

4. Solve the following equations referring to the preceding box.

a)  $x^2 = 9$   $S = \{-3, 3\}$       b)  $x^2 = 5$   $S = \{-\sqrt{5}, \sqrt{5}\}$       c)  $x^2 = -4$   $S = \emptyset$

5. Write the following equations in the form  $x^2 = k$  and then solve them.

a)  $x^2 - 16 = 0$   $S = \{-4, 4\}$       b)  $x^2 + 9 = 0$   $S = \emptyset$   
 c)  $x^2 - 2 = 0$   $S = \{-\sqrt{2}, \sqrt{2}\}$       d)  $2x^2 - 18 = 0$   $S = \{-3, 3\}$   
 e)  $3x^2 - 15 = 0$   $S = \{-\sqrt{5}, \sqrt{5}\}$       f)  $2x^2 + 50 = 0$   $S = \emptyset$   
 g)  $4x^2 - 9 = 0$   $S = \{-\frac{3}{2}, \frac{3}{2}\}$       h)  $2x^2 - 1 = 0$   $S = \{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\}$

## ACTIVITY 4 Form $a(x - h)^2 + k = 0$

a) Justify the steps in the solving of the equation  $(x - 3)^2 - 16 = 0$ .

$(x - 3)^2 - 16 = 0$

Add 16 to each side.

$\Leftrightarrow (x - 3)^2 = 16$

Apply the form  $x^2 = k$ .

$\Leftrightarrow x - 3 = -4$  or  $x - 3 = 4$

Solve each 1st degree equation.

$\Leftrightarrow x = -1$  or  $x = 7$

Establish the solution set.

Thus,  $S = \{-1, 7\}$ .

b) Justify the steps in the solving of the equation  $2(x + 1)^2 - 18 = 0$ .

$2(x + 1)^2 - 18 = 0$

Add 18 to each side.

$\Leftrightarrow 2(x + 1)^2 = 18$

Divide each side by 2.

$\Leftrightarrow (x + 1)^2 = 9$

Apply the form  $x^2 = k$ .

$\Leftrightarrow x + 1 = -3$  or  $x + 1 = 3$

Solve each 1st degree equation.

$\Leftrightarrow x = -4$  or  $x = 2$

Establish the solution set.

Thus,  $S = \{-4, 2\}$ .

- c) Justify the steps enabling you to conclude that the following equation has no solution.

$$3(x+1)^2 + 12 = 0$$

*Subtract 12 from each side.*

$$\Leftrightarrow 3(x+1)^2 = -12$$

*Divide each side by 3.*

$$\Leftrightarrow (x+1)^2 = -4$$

*The square of a real number cannot be negative.*

Thus  $S = \emptyset$ .

### FORM $a(x-h)^2 + k = 0$

- We use the following method consisting of isolating the variable to solve such an equation.

$$2(x+1)^2 - 10 = 0$$

$$\Leftrightarrow 2(x+1)^2 = 10$$

Add 10 to each side.

$$\Leftrightarrow (x+1)^2 = 5$$

Divide each side by 2.

$$\Leftrightarrow x+1 = -\sqrt{5} \quad \text{or} \quad x+1 = \sqrt{5}$$

Apply the form  $x^2 = k$ .

$$\Leftrightarrow x = -1 - \sqrt{5} \quad \text{or} \quad x = -1 + \sqrt{5}$$

Solve each 1st degree equation.

$$\text{Thus, } S = \{-1 - \sqrt{5}, -1 + \sqrt{5}\}.$$

Establish the solution set S.

- Note that the equation  $a(x-h)^2 + k = 0$  has no solution if  $a$  and  $k$  have the same sign.

6. Solve the following equations by isolating the variable.

a)  $(x-3)^2 = 16$   $S = \{-1, 7\}$

b)  $(2x+1)^2 = 9$   $S = \{-2, 1\}$

c)  $(2x-3)^2 - 1 = 0$   $S = \{1, 2\}$

d)  $2(x+1)^2 - 8 = 0$   $S = \{-3, 1\}$

e)  $-2(x-1)^2 + 18 = 0$   $S = \{-2, 4\}$

7. Show that the equation  $a(x-h)^2 + k = 0$  has the solution set

a)  $S = \left\{ h - \sqrt{\frac{-k}{a}}, h + \sqrt{\frac{-k}{a}} \right\}$  if  $\frac{-k}{a} > 0$ .

$$\begin{aligned} a(x-h)^2 + k = 0 &\Leftrightarrow a(x-h)^2 = -k \Leftrightarrow (x-h)^2 = \frac{-k}{a} \\ \Leftrightarrow x-h = -\sqrt{\frac{-k}{a}} \text{ or } x-h = +\sqrt{\frac{-k}{a}} &\Leftrightarrow x = h - \sqrt{\frac{-k}{a}} \text{ or } x = h + \sqrt{\frac{-k}{a}} \end{aligned}$$

b)  $S = \{h\}$  if  $k = 0$ .

*The solutions obtained in a) are equal to h when  $k = 0$ .*

c)  $S = \emptyset$  if  $\frac{-k}{a} < 0$ .  $\sqrt{\frac{-k}{a}} \notin \mathbb{R}$  when  $\frac{-k}{a} < 0$ . Therefore, no real solutions exist.

### SOLVING THE EQUATION $a(x-h)^2 + k = 0$

The equation  $a(x-h)^2 + k = 0$  has the solution set

- $S = \left\{ h - \sqrt{\frac{-k}{a}}, h + \sqrt{\frac{-k}{a}} \right\}$  if  $\frac{-k}{a} > 0$ .

- $S = \{h\}$  if  $k = 0$ .

- $S = \emptyset$  if  $\frac{-k}{a} < 0$ .

8. Solve the following equations using the formulas stated in the theory box on page 32.

a)  $(x-1)^2 + 9 = 0$   $S = \emptyset$       b)  $2(x-3)^2 = 0$   $S = \{3\}$   
c)  $(2x-5)^2 - 25 = 0$   $S = \{0, 5\}$       d)  $2(x+1)^2 - 14 = 0$   $S = \{-1-\sqrt{7}, -1+\sqrt{7}\}$   
e)  $-2(2x+1)^2 + 32 = 0$   $S = \left\{\frac{5}{2}, -\frac{3}{2}\right\}$       f)  $3(x-2)^2 + 27 = 0$   $S = \emptyset$

### SOLVING QUADRATIC EQUATIONS: THE DISCRIMINANT METHOD

- The discriminant, noted  $\Delta$  (read delta), of the quadratic equation  $ax^2 + bx + c = 0$  is the real number:

$$\Delta = b^2 - 4ac$$

- The existence and number of real solutions depend on the sign of the discriminant  $\Delta$ .

Sign of $\Delta$	Number of solutions	Solutions
$\Delta > 0$	2 solutions	$x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$
$\Delta = 0$	1 solution	$x = -\frac{b}{2a}$
$\Delta < 0$	no real solution	

$3x^2 - 11x - 4 = 0$   
•  $a = 3, b = -11, c = -4$   
•  $\Delta = b^2 - 4ac$   
 $\Delta = (-11)^2 - 4 \times 3 \times -4$   
 $\Delta = 169$   
•  $x_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{11-13}{6} = -\frac{1}{3}$   
 $x_2 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{11+13}{6} = 4$   
Thus,  $S = \left\{-\frac{1}{3}, 4\right\}$ .

$x^2 - 6x + 9 = 0$   
•  $a = 1, b = -6, c = 9$   
•  $\Delta = b^2 - 4ac$   
 $\Delta = (-6)^2 - 4 \times 1 \times 9$   
 $\Delta = 0$   
•  $x = -\frac{b}{2a} = \frac{6}{2} = 3$   
Thus,  $S = \{3\}$ .

$x^2 - x + 1 = 0$   
•  $a = 1, b = -1, c = 1$   
•  $\Delta = b^2 - 4ac$   
 $\Delta = (-1)^2 - 4 \times 1 \times 1$   
 $\Delta = -3$   
Thus,  $S = \emptyset$ .

9. Using the sign of the discriminant, indicate the number of solutions to the following equations.

a)  $2x^2 + 3x - 2 = 0$   $\Delta = 25; 2 \text{ solutions}$       b)  $-2x^2 - 5x + 3 = 0$   $\Delta = 49, 2 \text{ solutions}$   
c)  $4x^2 + 12x + 9 = 0$   $\Delta = 0; 1 \text{ solution}$       d)  $-x^2 + x - 1 = 0$   $\Delta = -3; 0 \text{ solution}$   
e)  $x^2 - 6x = 0$   $\Delta = 36; 2 \text{ solutions}$       f)  $2x^2 - 8 = 0$   $\Delta = 64; 2 \text{ solutions}$   
g)  $x(x-3) = -2$   $\Delta = 1; 2 \text{ solutions}$       h)  $-x^2 + 2x - 1 = 0$   $\Delta = 0; 1 \text{ solution}$

10. Solve the equations of the preceding exercise using the discriminant method.

a)  $S = \left[-2, \frac{1}{2}\right]$       b)  $S = \left[\frac{1}{2}, -3\right]$       c)  $S = \left[-\frac{3}{2}\right]$       d)  $S = \emptyset$   
e)  $S = \{0, 6\}$       f)  $S = \{-2, 2\}$       g)  $S = \{1, 2\}$       h)  $S = \{1\}$

**11.** Solve the following equations using the most appropriate method.

- a)  $2x^2 - x - 10 = 0$   $S = \left[-2, \frac{5}{2}\right]$       b)  $x^2 - 8x + 15 = 0$   $S = \{3, 5\}$   
c)  $x^2 = x$   $S = \{0, 1\}$       d)  $x^2 = 9$   $S = \{-3, 3\}$   
e)  $2x^2 + 5x = 0$   $S = \left\{0, -\frac{5}{2}\right\}$       f)  $x^2 - 6x = -5$   $S = \{1, 5\}$   
g)  $(2x + 1)^2 = 9$   $S = \{-2, 1\}$       h)  $\frac{3}{2}x^2 - 9x + 12 = 0$   $S = \{2, 4\}$   
i)  $3(x - 1)^2 - 12 = 0$   $S = \{-1, 3\}$       j)  $-2(x + 1)^2 + 6 = 0$   $S = \{-1 - \sqrt{3}, -1 + \sqrt{3}\}$

**12.** Show that when the solutions to the quadratic equation  $ax^2 + bx + c = 0$  exist,

- a) the sum of the solutions is equal to  $-\frac{b}{a}$ .      b) the product of the solutions is equal to  $\frac{c}{a}$ .
- $$\frac{-b - \sqrt{\Delta}}{2a} + \frac{-b + \sqrt{\Delta}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$
- $$\left(\frac{-b - \sqrt{\Delta}}{2a}\right)\left(\frac{-b + \sqrt{\Delta}}{2a}\right) = \frac{b^2 - \Delta}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

**13.** Solve the following equations.

- a)  $(2x + 1)^2 = (x - 3)^2$   $S = \left\{-4, \frac{2}{3}\right\}$       b)  $(2x + 3)(x - 2) = 9$   $S = \left\{-\frac{5}{2}, 3\right\}$   
c)  $x^2 + 2x - 1 = 0$   $S = \{-1 - \sqrt{2}, -1 + \sqrt{2}\}$       d)  $x^3 + 2x^2 - 3x = 0$   $S = \{0, 1, -3\}$

**14.** The length of a rectangular field measures 5 m more than twice its width. If the total area is equal to  $250 \text{ m}^2$ , what is the perimeter of the field? **70 m**

**15.** The height  $h(t)$  of a projectile, measured from ground level, is given by  $h(t) = -t^2 + 8t$ , where  $t$  represents the elapsed time in seconds since it was launched.

- a) Can the projectile hit a target located at a height of 20 m?  
**No, the equation  $-t^2 + 8t = 20$  has no solution, since  $\Delta = -16$ .**  
b) At what instant does the projectile hit the ground? **At  $t = 8 \text{ s}$ .**  
c) At what instant, during the projectile's ascent, is a target hit if it is located 15 m above the ground? **At  $t = 3 \text{ s}$ .**

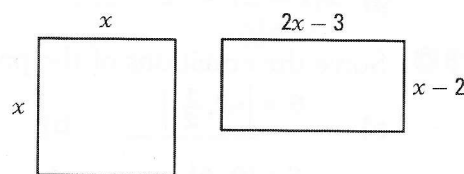
**16.** The value  $v(t)$ , in cents, of a share is given by  $v(t) = 2t^2 - 16t + 40$  where  $t$  represents the number of weeks since the share's purchase.

- a) After how many weeks is the share worth 58 ¢? **After 9 weeks**  
b) Can the share reach a value of 6 ¢? Justify your answer.  
**No, the equation  $2t^2 - 16t + 34 = 0$  has no solution since  $\Delta = -16$ .**

**17.** A mother is presently 5 years older than twice her daughter's age. Ten years ago, the product of the mother and daughter's ages was equal to 125. What is the mother's present age?  
**35 years old**

**18.** The square and the rectangle on the right have the same area. What is the numerical value of the rectangle's perimeter?

**$x = 6$ ; Perimeter = 26 units**



- 19.** In a triangle, the height relative to its base is 2 cm less than that base. Determine this height if the area of the triangle is equal to  $12 \text{ cm}^2$ .

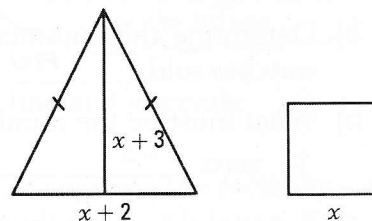
**Height: 4 cm**

- 20.** The height of a cylinder measures 4 cm more than its radius. Find the height of this cylinder if its total area is equal to  $140 \pi \text{ cm}^2$ .

**Height: 9 cm**

- 21.** An isosceles triangle and a square have the same area.  
What is the numerical value of the triangle's perimeter?

**27.7 u**



- 22.** Consider the figure on the right. The area of square EBGF is represented by the polynomial  $x^2 + 8x + 16$ .

If the area of rectangle ABCD is represented by the polynomial  $2x^2 + 13x - 7$ , what is the length of the rectangle?

**$2x - 1$**

