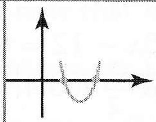
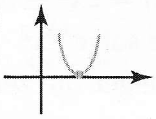
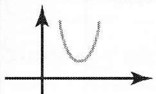
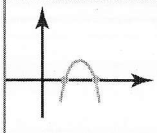
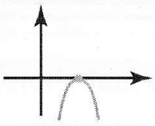
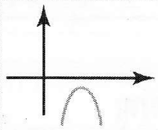
## GRAPHING A PARABOLA – GENERAL FORM

**Procedure**

1. Identify the parameters $a$, $b$ and $c$.
2. Determine the opening according to the sign of $a$.
3. Determine the coordinates of the vertex V.
   $V = \left(-\frac{b}{2a}, -\frac{\Delta}{2a}\right)$ where $\Delta = b^2 - 4ac$.
4. Find the zeros.
   $x_1 = \frac{-b - \sqrt{\Delta}}{2a}$; $x_2 = \frac{-b + \sqrt{\Delta}}{2a}$
5. Find the $y$-intercept.
6. Complete a table of values.
7. Graph the parabola.

We observe 6 possible situations according to the signs of $a$ and $\Delta$.

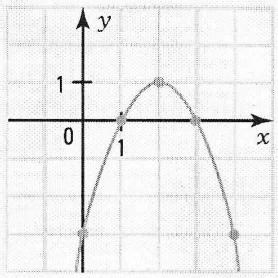
| | $\Delta > 0$ | $\Delta = 0$ | $\Delta < 0$ |
|---|---|---|---|
| $a > 0$ |  |  |  |
| $a < 0$ |  |  |  |

Ex.: $f(x) = -x^2 + 4x - 3$

1. $a = -1$, $b = 4$, $c = -3$
2. Open downward since $a < 0$
3. V(2, 1).
4. $\Delta = 4$. There are 2 zeros: $x_1 = 1$ and $x_2 = 3$.
5. $f(0) = -3$.
6. 

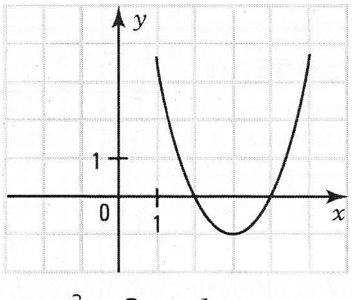
| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | –3 | 0 | 1 | 0 | –3 |

7. 

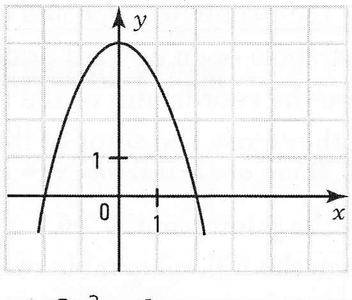



**2.** Graph the following parabolas.

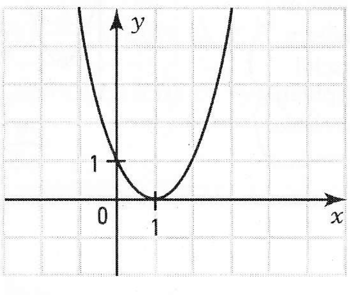
a) $y = x^2 - 6x + 8$



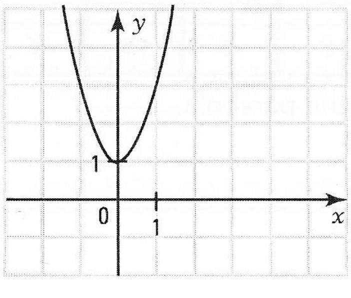

b) $y = -x^2 + 4$




c) $y = x^2 - 2x + 1$



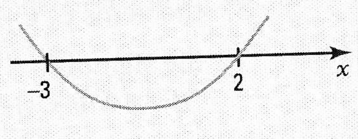

d) $y = 2x^2 + 1$



## SIGN OF A QUADRATIC FUNCTION – GRAPHICAL METHOD

To determine the sign of the quadratic function $f(x) = x^2 + x - 6$,

1. we determine the zeros of the function which are then placed on a number line.
2. we draw a sketch of the parabola taking into account its opening which depends on the sign of $a$.
3. we deduce the sign of the function.
   $f(x) \geq 0$ if $x \in ]-\infty, -3] \cup [2, +\infty[$. $f(x) \leq 0$ if $x \in [-3, 2]$.




**4.** Determine the sign of the following quadratic functions.

a) $f(x) = x^2 + 2x - 15$ ***$f(x) \geq 0$ if $x \in ]-\infty, -5] \cup [3, +\infty[$; $f(x) \leq 0$, if $x \in [-5, 3]$***

b) $f(x) = -2x^2 + 7x - 6$ ***$f(x) \geq 0$ if $x \in \left[\frac{3}{2}, 2\right]$; $f(x) \leq 0$ if $x \in \left]-\infty, \frac{3}{2}\right] \cup [2, +\infty[$***

c) $f(x) = x^2 - 2x + 1$ ***$f(x) \geq 0$, $\forall x \in \mathbb{R}$***

d) $f(x) = -4x^2 + 4x - 1$ ***$f(x) \leq 0$, $\forall x \in \mathbb{R}$***

**5.** Determine the domain and range of the following functions.

a) $f(x) = -x^2 + 4x + 5$ ***Dom $f = \mathbb{R}$; ran $f = ]-\infty, 9]$***

b) $f(x) = x^2 + 2x - 15$ ***Dom $f = \mathbb{R}$; ran $f = [-16, +\infty[$***

**6.** Study the variation of the following functions.

a) $f(x) = x^2 - x - 6$ ***$f \searrow$ if $x \in \left]-\infty, \frac{1}{2}\right]$ and $f \nearrow$ if $x \in \left[\frac{1}{2}, +\infty\right[$***

b) $f(x) = -2x^2 + 3x - 1$ ***$f \nearrow$ if $x \in \left]-\infty, \frac{3}{4}\right]$ and $f \searrow$ if $x \in \left[\frac{3}{4}, +\infty\right[$***

**7.** What are the zeros of the function $y = -3x^2 + 11x - 6$? ***$\frac{2}{3}$ and 3***

**8.** Find the values of $x$ for which $f(x) = x^2 + 5x - 14$ is positive. ***$]-\infty, -7] \cup [2, +\infty[$***

**9.** What is the range of the function $f(x) = -x^2 + 2x + 15$? ***Ran $f = ]-\infty, 16]$***

**10.** What is the $y$-intercept of $y = 3x^2 - 2x + 5$? ***5***

**11.** Find the extrema and its nature (maximum or minimum) for $y = -x^2 - 2x + 3$.

***A maximum; 4***

**12.** What is the equation of the axis of symmetry for the parabola $y = -2x^2 + 5x - 3$?

***The line with equation $x = \frac{5}{4}$***

**13.** For what values of $x$ is the function $f(x) = 2x^2 - x - 6$ decreasing?

***$x \in \left]-\infty, \frac{1}{4}\right]$***

# 3.6 Quadratic functions – Factored form

## ACTIVITY 1 Quadratic function – Factored form

Consider the quadratic function $f(x) = 2x^2 - 7x + 3$.

a) What is the value of parameter $a$? $a = 2$

b) Determine the zeros $x_1$ and $x_2$ of the function. $x_1 = \frac{1}{2}$ and $x_2 = 3$

c) The factored form of the quadratic function is $f(x) = a(x - x_1)(x - x_2)$.

Determine the factored form of $f(x) = 2x^2 - 7x + 3$. $f(x) = 2\left(x - \frac{1}{2}\right)(x - 3)$

d) Expand the factored form to get back to the general form.

$2\left(x - \frac{1}{2}\right)(x - 3) = (2x - 1)(x - 3) = 2x^2 - 7x + 3$

### QUADRATIC FUNCTION – FACTORED FORM

- Given the general form of a quadratic function $f(x) = ax^2 + bx + c$ with zeros $x_1$ and $x_2$. The factored form of the quadratic function is:

$$f(x) = a(x - x_1)(x - x_2)$$

Ex.: – $f(x) = -2x^2 + 5x - 3$ yields the zeros: $x_1 = \frac{3}{2}$ and $x_2 = 1$.

The factored form of $f$ is $f(x) = -2\left(x - \frac{3}{2}\right)(x - 1)$.

– $f(x) = 4x^2 - 12x + 9$ yields only one zero or two equal zeros: $x_1 = x_2 = \frac{3}{2}$.

The factored form of $f$ is $f(x) = 4\left(x - \frac{3}{2}\right)\left(x - \frac{3}{2}\right) = 4\left(x - \frac{3}{2}\right)^2$.

**1.** In each of the following cases, determine the factored form of the function.

a) $f(x) = 3x^2 - 5x - 2$ $f(x) = 3\left(x + \frac{1}{3}\right)(x - 2)$

b) $f(x) = 2x^2 + 7x + 6$ $f(x) = 2\left(x + \frac{3}{2}\right)(x + 2)$

c) $f(x) = x^2 - 8x + 15$ $f(x) = (x - 3)(x - 5)$

d) $f(x) = -2x^2 + x + 3$ $f(x) = -2\left(x - \frac{3}{2}\right)(x + 1)$

e) $f(x) = 4x^2 - 4x + 1$ $f(x) = 4\left(x - \frac{1}{2}\right)^2$

**2.** Consider the three forms of a quadratic function: $f(x) = a(x - h)^2 + k$ (standard form). $f(x) = ax^2 + bx + c$, (general form) and $f(x) = a(x - x_1)(x - x_2)$ (factored form).
For each given form, determine the two other forms.

a) $f(x) = 2(x - 1)^2 - 8$

$f(x) = 2x^2 - 4x - 6$ ***(general form)***

$f(x) = 2(x + 1)(x - 3)$ ***(factored form)***

b) $f(x) = x^2 - 10x + 16$

$f(x) = (x - 5)^2 - 9$ ***(standard form)***

$f(x) = (x - 2)(x - 8)$ ***(factored form)***

c) $f(x) = 4x^2 - 8x + 3$

$f(x) = 4(x - 1)^2 - 1$ ***(standard form)***

$f(x) = 4\left(x - \frac{3}{2}\right)\left(x - \frac{1}{2}\right)$ ***(factored form)***

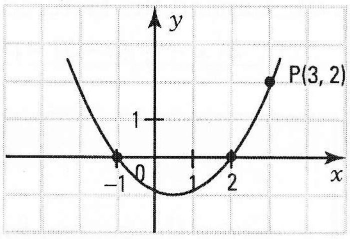
d) $f(x) = 2(x - 1)(x - 5)$

$f(x) = 2x^2 - 12x + 10$ ***(general form)***

$f(x) = 2(x - 3)^2 - 8$ ***(standard form)***

## ACTIVITY 2 Finding the rule – Given the zeros and a point

The parabola on the right has two zeros: –1 and 2 and passes through the point P(3, 2).



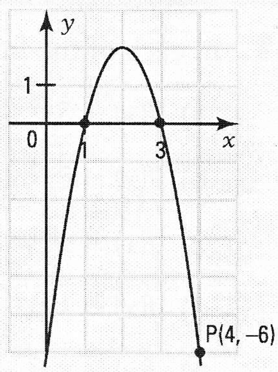

The quadratic function represented by this parabola has the rule:
$y = a(x - x_1)(x - x_2)$ (factored form).
Use the following procedure to determine the factored form of the rule.

1. Identify $x_1$ and $x_2$. $x_1 = -1$ ***and*** $x_2 = 2$
2. Determine $a$ knowing the coordinates of the point (3, 2) verify the rule.
   ***We have:*** $y = a(x + 1)(x - 2)$
   $2 = a(3 + 1)(3 - 2)$
   $2 = 4a$
   $a = \frac{1}{2}$
3. What is therefore the factored form of the rule? $y = \frac{1}{2}(x + 1)(x - 2)$
4. What is the general form? $y = \frac{1}{2}x^2 - \frac{1}{2}x - 1$

### FINDING THE RULE – GIVEN THE ZEROS AND A POINT

$$y = a(x - x_1)(x - x_2)$$

| | |
|---|---|
| **1.** Identify the zeros $x_1$ and $x_2$. | 1. $x_1 = 1$; $x_2 = 3$ <br> $y = a(x - 1)(x - 3)$ |
| **2.** Determine $a$ after replacing $x$ and $y$ in the rule by the coordinates of the point P. | 2. $-6 = a(4 - 1)(4 - 3)$ <br> $-6 = 3a$ <br> $a = -2$ |
| **3.** Deduce the rule of the function. | 3. $y = -2(x - 1)(x - 3)$ (factored form) <br> $y = -2x^2 + 8x - 6$ (general form) |



**3.** Find the rule, in general form, of each of the following quadratic functions.

a) A function with –5 and 2 as zeros and passing through the point P(3, 16).

$y = 2x^2 + 6x - 20$

b) A function with –3 and –1 as zeros and an initial value of –6.

$y = -2x^2 - 8x - 6$

c) A function with the unique zero –2 and passing through the point P(–1, 3).

$y = 3x^2 + 12x + 12$

d) A function with the vertex V(–1, 4) and passing through the point P(2, –5).

$y = -x^2 - 2x + 3$

e) A function with the vertex (1, –8) and one of the zeros equal to 3.

$y = 2x^2 - 4x - 6$

**4.** What is the vertex of the parabola that has –2 and 4 for zeros and passes through the point A(5, 21)?

$y = 3(x + 2)(x - 4)$; $V(1, -27)$

**5.** A parabola with zeros –1 and 3 passes through the point A(2, 6). What is the $y$-coordinate of the point B on the parabola that has an $x$-coordinate of 4?

$y = -2(x + 1)(x - 3)$; $B(4, -10)$. *The y-coordinate of point B is –10.*

**6.** A parabola with zeros –3 and 4 passes through the point A(2, –20). What are the points on this parabola that have a $y$-coordinate equal to 16?

$y = 2(x + 3)(x - 4)$; $P_1(-4, 16)$ and $P_2(5, 16)$

**7.** What is the $y$-intercept of the parabola with zeros –3 and –1 and passing through the point A(–2, 2)?

*The y-intercept is equal to –6.*

**8.** What is the equation of the axis of symmetry of a parabola with zeros –3 and 4?

$x = \frac{1}{2}$

**9.** The table of values on the right gives the coordinates of different points on a parabola. What is the equation of this parabola?

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| 1 | 8 |
| 3 | 8 |
| 5 | 0 |

*Axis of symmetry:* $x = 2$; *The zeros are –1 and 5.*

$y = -(x + 1)(x - 5)$; $y = -x^2 + 4x + 5$

**10.** Determine the range of the quadratic function $f$ with zeros 3 and 5 which verifies $f(2) = -6$.

$f(x) = -2x^2 + 16x - 30$; $V(4, 2)$; $\text{ran } f = ]-\infty, 2]$

**11.** What is the rule of the function $f$ that has a range of $]-\infty, 4]$ and is positive over the interval $[-1, 3]$? $f(x) = -x^2 + 2x + 3$

**12.** The value of a share, in dollars, $x$ weeks after its purchase is given by the rule $y = -0.1x^2 + x + 4.5$. Do you make a profit or a loss if the share is sold two weeks after reaching its maximum value?

***Value at purchase: \$4.50; V(5, 7); f(7) = \$6.60. A profit of \$2.10 per share is made.***

**13.** The position $f(t)$, in metres, of a diver relative to the surface is described by the rule $f(t) = 0.5t^2 - 6t + 10$ where $t$ represents elapsed time, in seconds. How long was the diver under water?

***$f(t) \leqslant 0 \Leftrightarrow 2 \leqslant t \leqslant 10$. The diver was under water during 8 seconds.***

**14.** The trajectory of a stone thrown from a seaside cliff is a partial parabola. The position $f(t)$, in metres, of the stone relative to sea level is given by $f(t) = -t^2 + 8t + 20$ where $t$ represents elapsed time in seconds since it was thrown. How many seconds after reaching its maximum height will the stone hit the water?

***After 6 seconds.***

**15.** The manager of a movie theatre has calculated the following results. When the cost of admission is set at \$10, he observes on average 100 spectators per showing and for each \$0.50 rebate on the admission price, he notices an average of 10 more spectators.

a) Find the rule which gives the total revenue per showing as a function of the number $x$ of \$0.50 rebates.

***R(x): Total revenue per showing.***

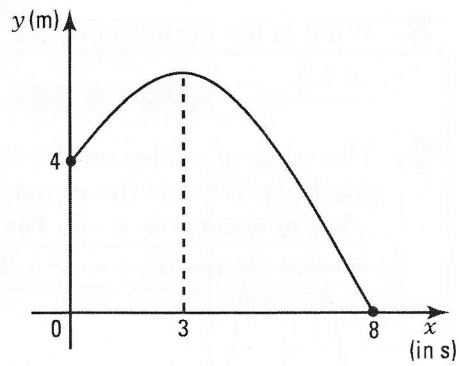
***$R(x) = (10 - 0.5x)(100 + 10x)$; $R(x) = -5x^2 + 50x + 1000$***

b) 1. At what amount should the manager set the cost of admission in order to maximize the revenue per showing?

***The function R reaches its maximum when x = 5. The price of admission should be set at \$7.50.***

2. What is the total maximum revenue per showing? ***\$1125***

**16.** A stone is thrown upward from a height of 4 m. After 3 s, it reaches its maximum height and after 8 s, it hits the ground. Its trajectory is parabolic.




1. What is the maximum height reached by the stone?

***6.25 m***

2. Determine the elapsed time from the moment the stone was at a height of 2.25 m during its descent to the moment it hit the ground.

***1 second***

**9.** The trajectory of a ball thrown by David is a partial parabola. The height $h(t)$, in metres, reached by the ball is described by the rule $h(t) = -(t - 3)^2 + 9$.

Determine over what interval of time the ball is at a height greater than or equal to 8 m above ground.

***[2, 4]***

**10.** A share purchased for $4 reaches its maximum value of $4.50 five weeks after its purchase. The function associating the value $v(t)$, in dollars, of the share as a function of time $t$, in weeks, has been shown to be a quadratic function. What is the value of the share 8 weeks after its purchase?

***$v(t) = -0.02(t - 5)^2 + 4.50$; $v(8) = 4.32$***

***The share is worth $4.32 eight weeks after its purchase.***

**11.** The number of units $q(x)$ produced per day by $x$ employees is given by the rule $q(x) = -0.25x^2 + 10x$ ($x \leq 25$).

a) What is the maximum number of units produced in one day? How many employees are required to produce this maximum?

***100 units produced by 20 employees.***

b) How many employees are required to produce 75 units?

***10 employees***

**12.** A truck with a height of 190 cm enters a tunnel with a parabolic ceiling. The width of the tunnel is 20 m and the maximum height of the tunnel is 10 m. At what minimal distance from the edge at ground level can this truck pass through the tunnel? ***1 m***

**13.** The rule $p = 1000 - 2q$ enables you to calculate the selling price $p$ of a parasol as a function of the number $q$ of parasols ordered. What must the number of parasols ordered be to maximize the revenue generated by the sale of the parasols? ***250 parasols***

**14.** A stone is thrown vertically upward. The function which gives the height $h(t)$, in metres, as a function of elapsed time $t$, in seconds, since the stone was thrown is a quadratic function with the rule: $h(t) = -2t^2 + 12t$.

a) 1. What is the maximum height reached by the stone? ***18 m***

2. At what time does the stone reach its maximum height? ***3 seconds***

b) At what time, during its descent, does the stone reach a height of 16 m?

***4 seconds after it was thrown.***

**15.** A rectangular yard is to be fenced in with 80 m of fence. What must the dimensions of the yard be in order to maximize the area of the field?

***x: width of the yard; 40 – x: length of the yard***

***A(x): area of the yard. A(x) = $-x^2 + 40x$***

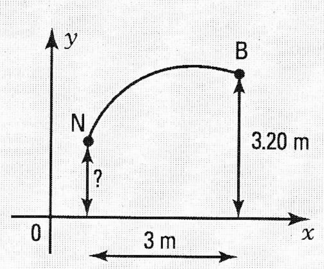
***The vertex of the parabola representing A(x) has the coordinates V(20, 400).***

***The yard must be in the shape of a square with 20 m sides.***

**16.** Nancy throws a ball toward a basket located 3.2 m off the ground. The ball's trajectory is represented on the right.

The rule associated with this trajectory is: $y = -0.4(x - 6)^2 + 3.6$.

Nancy throws the ball at a distance of 3 m from the basket. From what height did Nancy throw the ball?

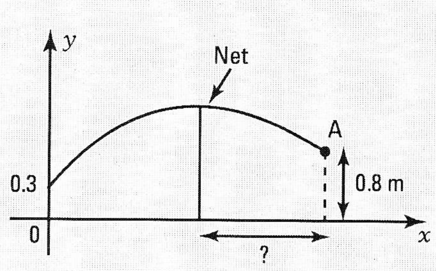



***$y_B = 3.20$; $x_B = 7$; $x_N = 4$; $y_N = 2$***

***The ball is thrown from a height of 2 m.***

**17.** During a tennis match, Karen hits the ball to Alex. The trajectory of the ball is represented in the Cartesian plane by a parabola with its vertex over the net. The equation of the trajectory is: $f(x) = -0.1(x - 3)^2 + 1.2$.

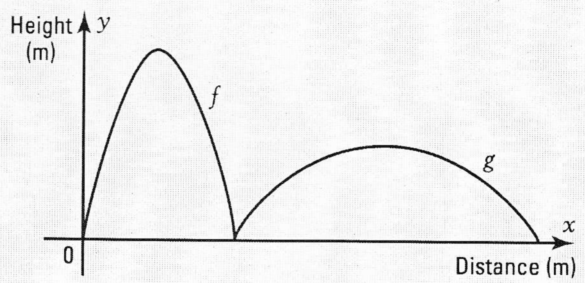
The ball is hit by Karen at a height of 0.3 m and reaches Alex at a height of 0.8 m on its descent. How far is Alex from the net if the vertex of the ball's trajectory is directly over the net?




***$x_A = 5$; $x_V = 3$; $x_A - x_V = 2$. Alex is located 2 m from the net.***

**18.** A kangaroo makes two consecutive jumps. The trajectory is represented by two portions of parabolas associated with the functions *f* and *g*.

The rule associated with the second jump is $g(x) = -0.25(x - 6.4)^2 + 2.56$. What is the rule associated with the first jump if the kangaroo jumped twice as high on the first jump as the second jump? (The variables *x* and *y* are expressed in metres.)




***The zeros of g are 3.2 and 9.6.***

***Vertex of the 1st parabola: V(1.6, 5.12); $f(x) = -2(x - 1.6)^2 + 5.12$.***