2.6 Optimization of a situation

Activity 1 Two-variable function

a) At a theater, floor seats cost \$12 and balcony seats cost \$16. Let x represent the number of floor seats sold and y the number of balcony seats sold.
Express the proceeds R of the theater as a function of the variables x and y. ***R = 12x + 16y***

b) A company produces tables and chairs. The production costs are \$150 per table and \$50 per chair. Let x and y represent respectively the number of tables and the number of chairs produced.
Express the production cost C as a function of the variables x and y. ***C = 150x + 50y***

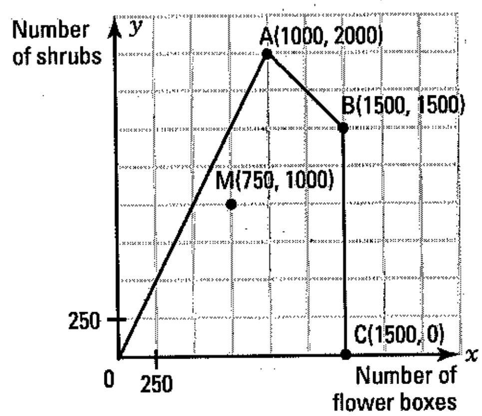
> **OPTIMIZATION OF A FUNCTION**
>
> **Optimizing** a function of two variables x and y consists in finding the couple (x, y) which, depending on context, **maximizes** or **minimizes** the function.
>
> In general, we seek the couple (x, y) which maximizes a revenue function or minimizes a cost function.

Activity 2 Maximization of a revenue function

At the end of the season, the manager of a nursery garden wants to clear his inventory which contains 1500 flower boxes and 2000 shrubs.

Let x and y represent respectively the number of flower boxes and the number of shrubs sold.

The constraints associated with the sale of the flower boxes and shrubs are represented by the polygon of constraints on the right. The revenue R (in \$) generated by selling x flower boxes and y shrubs is given by $R = 3x + 8y$.



a) The interior point M(750, 1000) of the polygon satisfies the constraints and corresponds to the sale of 750 flower boxes and 1000 shrubs. What is the revenue R generated by this sale?

R = 3 × 750 + 8 × 1000 = \$10 250

b) Evaluate, for each vertex of the polygon of constraints, the revenue associated with the sale.

Vertices	Revenue: $R = 3x + 8y$
0 (0, 0)	***R = 3 × 0 + 8 × 0 = \$0***
A (1000, 2000)	***R = 3 × 1000 + 8 × 2000 = \$19 000***
B (1500, 1500)	***R = 3 × 1500 + 8 × 1500 = \$16 500***
C (1500, 0)	***R = 3 × 1500 + 8 × 0 = \$4500***

c) 1. Among the vertices 0, A, B and C, which one corresponds to the maximal revenue? ***A***
What is this maximal revenue? ***$19 000***

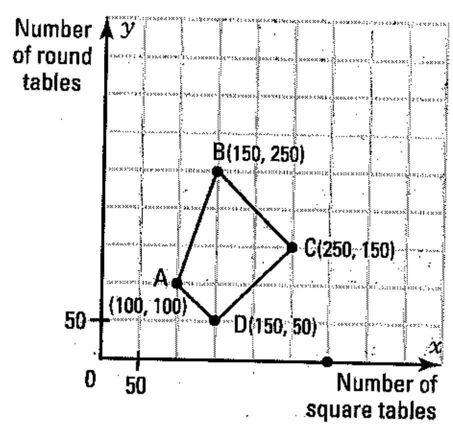
2. Among the vertices 0, A, B and C, which one corresponds to the minimal revenue? ***0***
What is this minimal revenue? ***$0***

d) 1. Choose a random point in the interior of the polygon of constraints. Evaluate the revenue associated with the chosen point and verify that the revenue obtained is contained between the minimal revenue and the maximal revenue computed in c).
Various answers.

2. What is the couple (x, y) verifying the constraints and which maximizes the revenue function? ***The couple (1000, 2000)***
Interpret your result. ***The manager of the nursery will maximize the revenue if he sells 1000 flower boxes and 2000 shrubs. His revenue will then be equal to $19 000.***

ACTIVITY 3 Minimization of a cost function

A company produces square tables and round tables. Let x and y represent respectively the number of square tables and the number of round tables. The constraints associated with the production of the tables are represented by the polygon of constraints on the right.



The cost C (in $) associated with the production of x square tables and y round tables is given by $C = 120x + 140y$.

a) Evaluate, for each vertex of the polygon of constraints, the cost associated to the production.

Vertices	Cost: $C = 120x + 140y$
A (100, 100)	***$26 000***
B (150, 250)	***$53 000***
C (250, 150)	***$51 000***
D (150, 50)	***$25 000***

b) 1. Among the vertices A, B, C and D, which one corresponds to the minimal cost? ***D*** What is this minimal cost? ***$25 000***

2. Among the vertices A, B, C and D, which one corresponds to the maximal cost? ***B***
What is this maximal cost? ***$53 000***

c) 1. Choose a random point in the interior of the polygon of constraints and verify that the cost obtained is contained between the minimal cost and the maximal cost established in b).
Various answers.

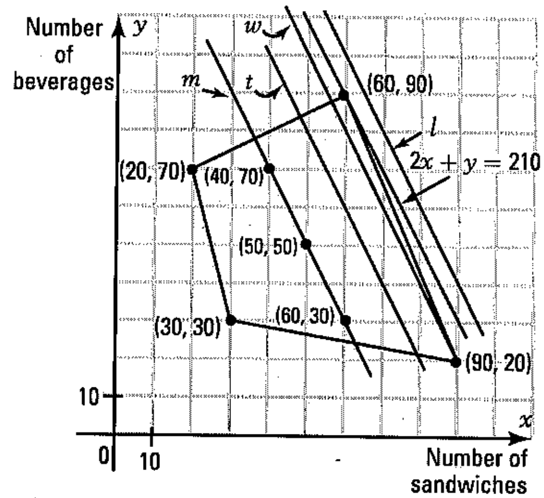
2. What is the couple (x, y), verifying the constraints, which minimizes the cost function?
The couple (150 , 50)
Interpret your result. ***The company will minimize its costs if it produces 150 square tables and 50 round tables. The cost will then be equal to $25 000.***

ACTIVITY 4 The scanning line

At lunchtime, a cafeteria sells sandwiches and beverages. Let x and y represent respectively the number of sandwiches sold and the number of beverages sold.

The constraints associated with the sale of these items are represented by the polygon of constraints on the right.

The revenue R (in \$) generated by selling x sandwiches and y beverages is given by $R = 2x + y$.



a) On Monday, the cafeteria's revenue was \$150.

1. Name three possible couples (x, y) which result in this \$150 revenue and which satisfy the constraints. ***For example: 40 sandwiches and 70 beverages or 50 sandwiches and 50 beverages or 60 sandwiches and 30 beverages.***
2. The three couples obtained verify the equation of the line m: $2x + y = 150$. Draw line m and verify that it passes through the three points obtained in 1.

b) On Tuesday, the cafeteria's revenue was \$180.

The couples which result in a revenue of \$180 verify the equation of the line t: $2x + y = 180$.

1. Draw line t.
2. Give a point on line t which satisfies the constraints and one point which does not satisfy the constraints. ***(50, 80) satisfies the constraints, (40, 100) doesn't satisfy the constraints.***
3. Explain why lines m and t are parallel. ***They have the same slope.***

c) When we vary the revenue R in the equation $2x + y = R$, the line $2x + y = R$ moves in the same direction.

The line $2x + y = R$ is called **scanning line.**

The lines m: $2x + y = 150$ and t: $2x + y = 180$ are thus two positions for the scanning line.

On Wednesday, the cafeteria's revenue was \$200. Give the equation of the line w corresponding to Wednesday's revenue and draw the scanning line corresponding to Wednesday.

w: 2x + y = 200

d) 1. Draw the line l: $2x + y = 220$ corresponding to a position of the scanning line. Can we find, on this line l, a point (x, y) satisfying the constraints? Justify your answer.

No, because every point on the line is outside the polygon of constraints.

2. Then, is a revenue of \$220 possible? ***No***

e) 1. What is the point of the polygon of constraints through which the scanning line giving the maximal revenue must pass? ***The vertex (60, 90)***

2. What is the maximal revenue? ***\$210***
3. Draw in red the scanning line passing through the point found and then give the equation of the line passing through the point found.

2x + y = 210

OPTIMIZATION OF A FUNCTION UNDER CONSTRAINTS

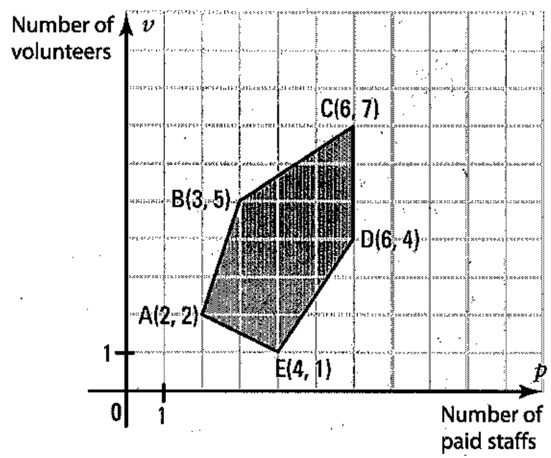
- Given a polygon of constraints and the function to be optimized $F = ax + by + c$, **optimizing** the function F consists in determining the points, if they exist, of the polygon of constraints whose coordinates **maximize** or **minimize**, depending on context, the function F.
- If the function to be optimized possesses a maximal or minimal value, then this value is attained at, at least, one of the vertices of the polygon of constraints.
- In practice, it is sufficient to evaluate the function to be optimized at each of the vertices of the polygon of constraints to establish the maximal (or minimal) value of the function to be optimized and thus deduce the couple which maximizes (or minimizes) the function.

1. The board of directors of a music competition hires paid staff and volunteers for its fundraising campaign. The constraints associated with the hiring process are represented by the polygon of constraints below.

The function F giving the costs (in $) associated with the hiring process is defined by $F = 120p + 50v$ where p and v represent respectively the number of paid staff and the number of volunteers.

a) Evaluate the function to be optimized for each vertex of the polygon of constraints.

Vertex	$F = 120p + 50v$
A(2, 2)	***$340***
B(3, 5)	***$610***
C(6, 7)	***$1070***
D(6, 4)	***$920***
E(4, 1)	***$530***



b) 1. Taking the constraints into account, what is the minimal cost associated with the hiring process for this fundraising campaign? ***$340***

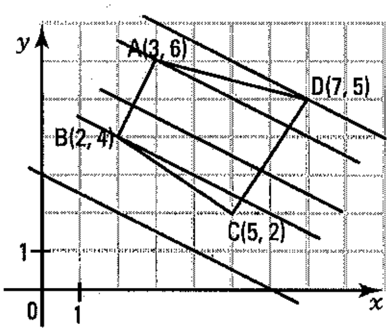
2. How many paid staffs and how many volunteers must the board hire in order to minimize costs? ***2 paid staffs and 2 volunteers.***

2. The vertices of a polygon of constraints are represented on the graph on the right.

The function to be optimized is defined by $R = 2x + 4y - 12$.

a) What is the couple which maximizes this function? ***(7, 5)***

b) Verify your result using the scanning line.

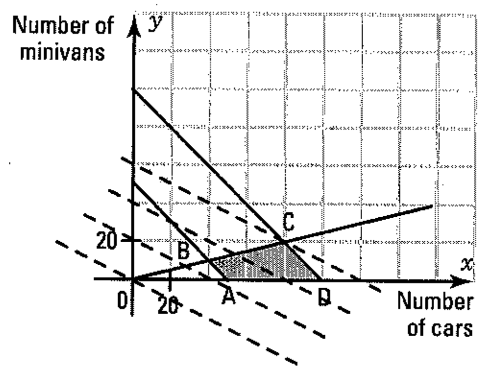


3. To raise money for their graduation party, secondary 5 students organize a car wash for cars and minivans.

The students charge \$5 per car and \$10 per minivan.

This event is organized with the following constraints.

- They can wash at most 100 vehicles.
- They must wash at least 50 vehicles in order to raise enough money.
- They expect to wash at least four times as many cars as minivans.



a) Identify the variables in this situation.

x: number of cars; y: number of minivans.

b) Translate the constraints into a system of inequalities.

$x \geq 0$

$y \geq 0$

$x + y \leq 100$

$x + y \geq 50$

$x \geq 4y$

c) Draw the polygon of constraints.

d) Establish the rule of the function to be optimized.

$R = 5x + 10y$

e) Evaluate the function to be optimized at each vertex of the polygon of constraints.

Vertices	$R = 5x + 10y$
A(50, 0)	250
B(40, 10)	300
C(80, 20)	600
D(100, 0)	500

f) How many cars and minivans must the students wash in order to maximize the profit? ***They must wash 80 cars and 20 minivans.***

g) Verify your result by drawing the scanning line and by making it vary.

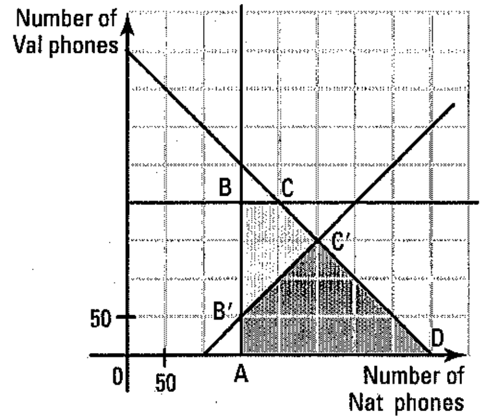
The last point of the polygon of constraints through which the scanning line passes is the point C(80, 20).

4. A cell phone company makes Nat model phones and Val model phones.

The company expects a revenue of $40 per Nat phone and $60 per Val phone.

To satisfy production constraints, the company must produce monthly,

- at most 400 cell phones;
- at least 150 Nat phones;
- at most 200 Val phones.



a) Identify the variables in this situation.

x: number of Nat phones; y: number of Val phones.

b) Translate the constraints into a system of inequalities.

$x \geqslant 0$

$y \geqslant 0$

$x + y \leqslant 400$

$x \geqslant 150$

$y \leqslant 200$

c) Draw the polygon of constraints.

d) Establish the rule of the function to be optimized.

R = 40x + 60y

e) Evaluate the function to be optimized at each vertex of the polygon of constraints.

Vertices	R = 40x + 60y
A(150, 0)	6000
B(150, 200)	18 000
C(200, 200)	20 000
D(400, 0)	16 000

f) How many cell phones of each model must the company make in order to maximize its revenue?

It must make 200 Nat phones and 200 Val phones.

g) After one week of production, the company decides to make at least 100 more Nat phones than Val phones.

Translate this additional constraint into an inequality. $x \geqslant y + 100$

h) Evaluate the function to be optimized at each vertex of the new polygon of constraints then determine the number of cell phones of each model the company must make in order to maximize its revenue.

Vertices	R = 40x + 60y
A′(150, 0)	6000
B′(150, 50)	9000
C′(250, 150)	19 000
D(400, 0)	16 000

It must make 250 Nat phones and 150 Val phones.

i) Did the maximal revenue increase or decrease as a result of this additional constraint?

It decreased by $1000.

SOLVING AN OPTIMIZATION PROBLEM

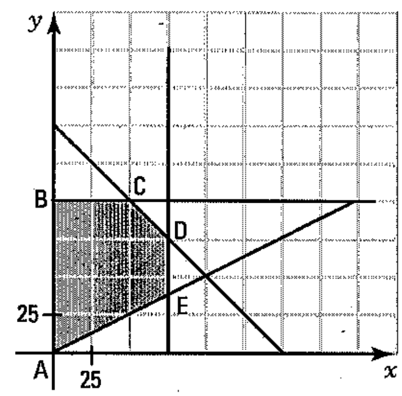
The following steps allow us to solve an optimization problem.

1. Identify the variables.
2. Translate the constraints of the situation into a system of inequalities.
3. Draw the polygon of constraints.
4. Determine the coordinates of the vertices of the polygon of constraints.
5. Establish the rule of the function to be optimized.
6. Evaluate the function to be optimized at each vertex of the polygon of constraints.
7. Deduce the vertex whose coordinates maximize (or minimize) the function to be optimized.

5. A landscape architect was hired by a cultural centre to design the exterior of the centre. The architect must observe the following constraints.

- The total area to be landscaped is at most 150 m^2.
- She must allot, at most, 75 m^2 for a flower bed and at most 100 m^2 for shrubs.
- She must allot, at most, an area twice as large for flowers as for shrubs.

Knowing that she charges \$200 per m^2 for flowers and \$125 per m^2 for shrubs, what area should she allot for each type of plant in order to maximize her revenue?



x: area allotted for flowers

y: area allotted for shrubs

$x \geqslant 0$

$y \geqslant 0$

$x + y \leqslant 150$

$x \leqslant 75$

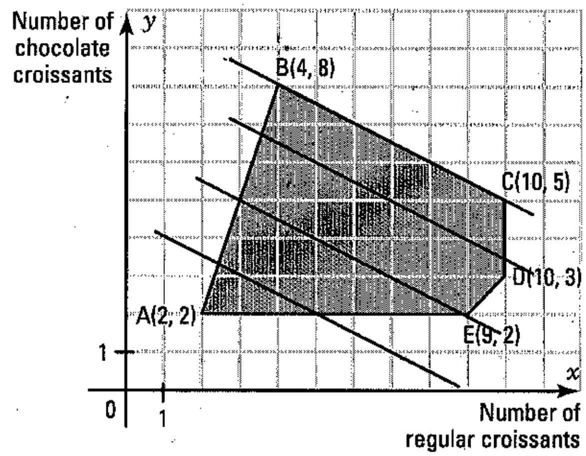
$y \leqslant 100$

$x \leqslant 2y$

Vertices	R = 200x + 125 y
A(0, 0)	0
B(0, 100)	12 500
C(50, 100)	22 500
D(75, 75)	24 375
E(75, 37,5)	19 687.50

She must allot 75 m² for flowers and 75 m² for shrubs.

ACTIVITY 3 Non unique optimal solution



The function $R = 1.25x + 2.50y$ gives the revenue of a baker who sells x regular croissants and y chocolate croissants.

The polygon of constraints is represented on the right.

a) Evaluate the revenue at each vertex of the polygon of constraints.

A: ***$7.50*** B: ***$25*** C: ***$25*** D: ***$20*** E: ***$16.25***

b) Verify that the revenue is maximal at the two consecutive vertices B and C of the polygon of constraints.

c) Draw a few positions for the scanning line and verify that the extreme position of the wandering line that verifies the constraints is the line BC.

d) What is the equation of line BC? ***1.25x + 2.50y = 25***

e) Is it true that any point on edge BC of the polygon of constraints whose coordinates are integers corresponds to a sale that maximizes the revenue? ***True***

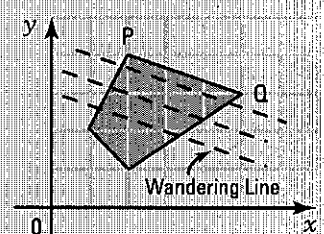
f) Give the 4 solution couples that maximize the revenue. ***(4, 8); (6, 7); (8, 6) and (10, 5).***

NON UNIQUE OPTIMAL SOLUTION

If a function to be optimized attains its maximal (or minimal) value at two consecutive vertices P and Q of a polygon of constraints, then this function attains this same maximal (or minimal) value at each point of the edge PQ of the polygon.

This situation occurs when the scanning line is parallel to one of the edges of the polygon of constraints.

Ex.: See activity 3.

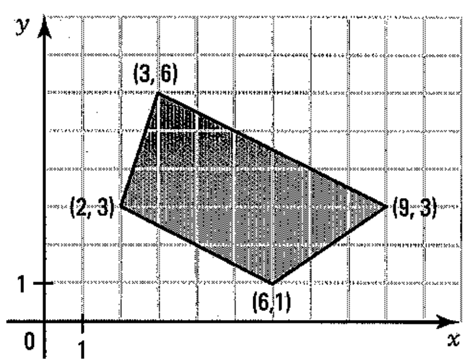


6. The polygon of constraints of an optimization problem is represented on the graph on the right.

The rule of the function to be optimized is: $R = 3x + 6y$ where x and y are integers.

How many couples maximize function R? ***4***

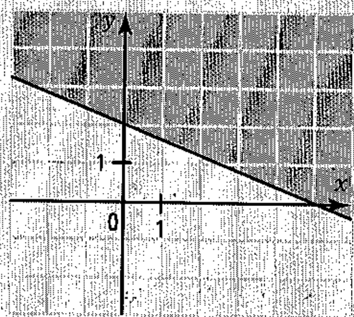
Name them: ***(3,6), (5,5), (7,4), (9,3)***



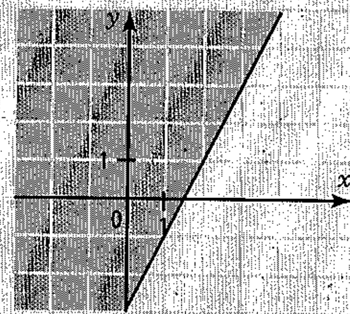
Evaluation 2

1. Represent graphically the solution set of the following inequalities.

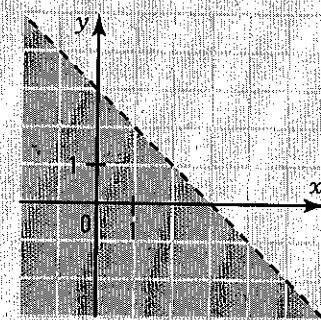
a) $2x + 5y \geqslant 10$



b) $y \geqslant 2x - 3$



c) $x + y < 3$



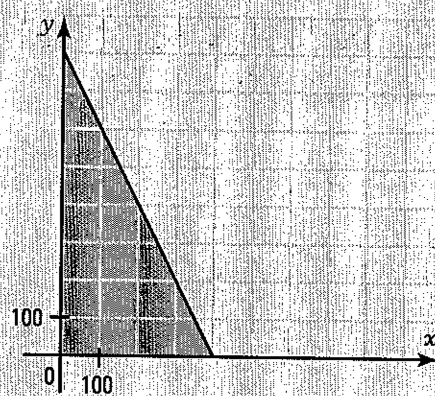
2. A bakery sells each day fresh bread and baguettes.

The price of a loaf of bread is \$3 and that of a baguette is \$1.50. In a day, the amount sold is at most equal to \$1200.

Represent this situation in the Cartesian plane.

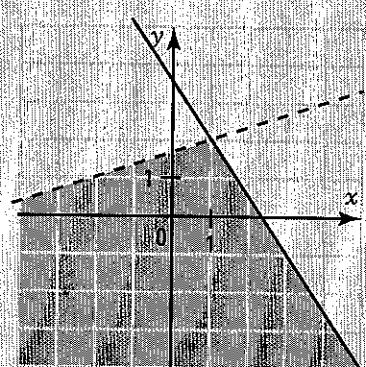
x: number of loaves of bread, y: number of baguettes.

$3x + 1.50y \leqslant 1200$.

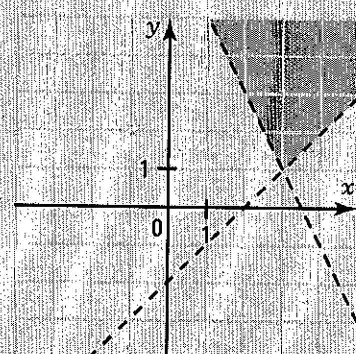


3. Represent graphically the solution set of the following systems of inequalities.

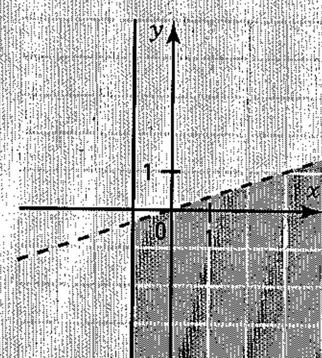
a) $\begin{cases} 3x + 2y \leqslant 7 \\ x - 3y > -5 \end{cases}$



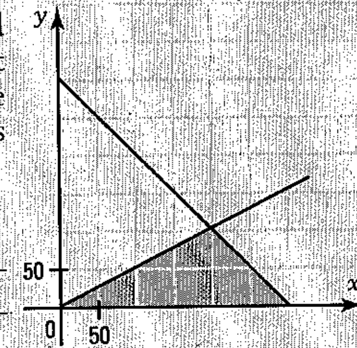
b) $\begin{cases} y > -2x + 7 \\ x - y < 2 \end{cases}$



c) $\begin{cases} x \geqslant -1 \\ x > 3y \end{cases}$



4. At a music camp, the main instruments taught are piano and violin. The camp managers expect that there will be at least twice as many campers playing piano as campers playing the violin. In addition, they expect a maximum of 300 campers playing one of these two instruments.



a) Represent this situation in the Cartesian plane.

***x*: number of campers playing piano,**

***y*: number of campers playing the violin;**

$x \geqslant 2y$

$x + y \leqslant 300$

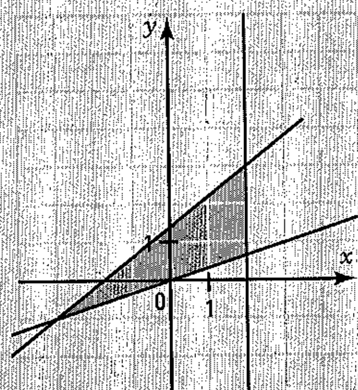
b) Give a solution couple of the system which satisfies these constraints.

(150, 50). 150 campers playing piano and 50 campers playing the violin.

5. Represent, for each of the following systems of inequalities, the polygon of constraints corresponding to the solution set of the system and then determine the polygon's vertices.

a) $-4x + 5y \leqslant 7$

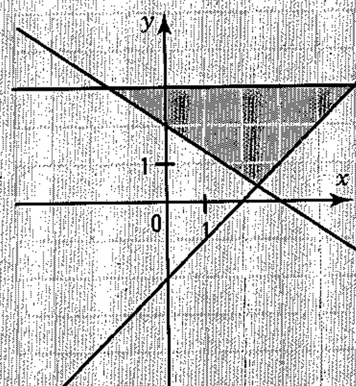
$x \leqslant 3y$
$x \leqslant 2$



$(-3, -1), (2, 3), \left(2, \frac{2}{3}\right)$

b) $2x + 3y \geqslant 6$

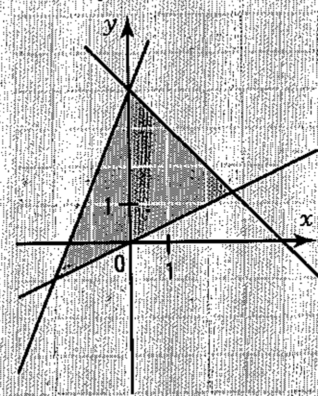
$x \leqslant y + 2$
$y \leqslant 3$



$\left(-\frac{3}{2}, 3\right), (5, 3), \left(\frac{12}{5}, \frac{2}{5}\right)$

c) $y \leqslant \frac{5}{2}x + 4$

$x + y \leqslant 4$
$x \leqslant 2y$



$(-2, -1), (0, 4), \left(\frac{8}{3}, \frac{4}{3}\right)$